CLASSICAL THEORETICAL PHYSICS (2017)

Linear Stability of Saturn's Rings

Sydney Terris
Department of Physics, Brown University, Providence, Rhode Island 02906, USA
(Received 5 December 2017)

Using a linear stability analysis of a system of *n* equal-mass bodies in circular orbit about a large primary mass, the compositional dynamics of Saturn's ring system is determined. Using a linear stability matrix and determining the associated eigenvalues, the inequality dictating the relationship between the cumulative mass of the *n* bodies and the mass of Saturn is derived. The results regain Maxwell's crucial theoretical conclusions on the stability of Saturn's rings, proving that a system of *n* non-interacting bodies in planar circular orbit exhibits stability, even against in-plane perturbations. Centrally, the derived mass ratio inequality relationship sufficiently predicts the many-body nature of Saturn's rings and aligns with spectroscopic and experimental data.

I. INTRODUCTION

While ring systems are generally highly collisional and unstable due to energy dissipation, Saturn's rings, closely representing an annular disk with concentric local maxima and minima, remain in stable orbit around the planet. The composition of Saturn's ring system remained a theoretical mystery until Maxwell's mathematical analysis of ring stability for the University of Cambridge Adam Prize in 1856, which determined that the rings are composed of isolated masses of varying size. As Maxwell states in his argument, "the only system of rings which can exist is one composed of an infinite number of unconnected particles revolving round the planet with different velocities according to their respective distances" [1]. That is to say, Maxwell eliminated the possibility of Saturn's ring structure as being either a rigid disk or a fluid system of gaseous particles [1].

Most importantly, a ring of equal-mass particles in circular orbit about a mass much larger than that of the particles remains in stable motion even when affected by small perturbation disturbances [2].

In this analysis, Saturn's ring system will be treated as a planar *n*-body problem and only in-plane perturbations will be considered and out-of-plane perturbations will be considered negligible. First, a block circulant matrix describing the linear approximation of the system dynamics and in-plane perturbations is constructed. Leveraging the determinant of the matrix, the eigenvalues are calculated. From the eigenvalues, an inequality is derived relating the mass of the primary, Saturn, to the cumulative mass of the *n* bodies in circular orbit about Saturn. The inequality is assessed in the limit of large *n* where the mass of each satellite body is vanishingly small in comparison with the mass of Saturn. From this limit, Maxwell's 1856 conclusion regarding the ratio of the ring mass to Saturn's mass is regained, defining the mass ratio conditions under which Saturn's ring system is stable.

II. THEORETICAL BACKGROUND

A linear stability analysis of a system of *n* equal-mass bodies in circular orbit about a very large mass provides insight into the mechanics of Saturn's ring system. In this mathematical analysis, Saturn's ring system will be treated as a planar *n*-body problem and only in-plane perturbations will be considered. As Maxwell concluded in his 1856 analysis, out-of-plane perturbations are less destabilizing than in-plane perturbations and, therefore, the out-of-plane perturbations have only negligible effects which can be ignored for simplicity [3]. Ultimately, this allows for the derivation to be performed in complex notation, clarifying the mathematical methods.

Notably, the system of Saturn's rings parallels n-body satellite systems of vanishingly small masses in rotation about a center of mass point. Moeckel's 1994 discussion of the linear stability of such systems, rooted in the invariant subspace of the linearized Hamiltonian, illustrates ring stability only at $n \geq 7$ [4].

To begin the discussion, Newton's law of gravitation defines dependence of the angular velocity of the particles on their radius r from Saturn, the large primary of interest. Introducing a small perturbation of the system will lead to a circulant matrix and the eigenvalues of that matrix define a linear approximation of the perturbation. Using this, Maxwell's results for cases of large n where n represents the small-body masses will be derived. From this, the conclusion is reached that Saturn's rings are stable so long as the following inequality is not violated:

$$m_{Rings} \leq \frac{2.298 M_{Saturn}}{n^2} \qquad (1)$$

[3].

III. METHOD

To consider n bodies in circular orbit about Saturn, the system must be counter-rotated so that all of the bodies remain at rest with the planet serving as the center of mass [3]. Then, the system is perturbed and the effect of this perturbation is analyzed to determine the nature of the ring system's stability condition. Even further, each ring body is repositioned to lie on the x-axis so that the perturbations in the real part represent radial perturbations, giving the stability matrix a circulant nature [3].

Importantly, the stability matrix is a block circulant matrix, meaning that it is a circulant matrix composed of small matrices, called blocks. The stability matrix for Saturn's ring system is composed of four blocks. Further, a circulant matrix is a matrix whose rows are cyclically shifted versions of a list such that each row is rotated one element to the right relative to the preceding row.

For the linear stability analysis of a system of n equal-mass bodies in circular orbit about a much more massive primary, Newton's law of gravitation yields a relationship between the angular velocity of the ring bodies and their radius from the center of mass. From this, the equation of motion for the system with respect to the angular velocity is determined and the system is perturbed in order to analyze first-order stability [3].

This is the starting point for this analysis' derivation of the mass ratio condition for stability. First, a block circulant matrix is employed to characterize the linear approximation of in-plane perturbations to the system. An analysis of the circulant matrix,

$$\frac{d}{dt}\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} \approx \left[\begin{array}{cccc|cccc} & & & & I & & & \\ & & & & & I & & \\ & & & & & & \ddots & \\ & & & & & & & I \\ \hline D & N_1 & \cdots & N_{n-1} & \Omega & & & \\ N_{n-1} & D & \cdots & N_{n-2} & & \Omega & & \\ \vdots & \vdots & \ddots & \vdots & & & \ddots & \\ N_1 & N_2 & \cdots & D & & & & \Omega \end{array}\right] \begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \\ \delta \dot{W}_0 \\ \delta \dot{W}_1 \\ \vdots \\ \delta \dot{W}_{n-1} \end{bmatrix} \qquad (2)$$

described in Vanderbei and Kolemen's *Linear Stability of Ring Systems* roots the mathematical investigation [3].

The complex matrices D, Ω, and N_k, used to simplify the block circulant matrix representation, are denoted as

$$D = \frac{3}{2}\omega^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{Gm}{2r^3}\begin{bmatrix} 1 - I_n + J_n/2 & 3 - 3J_n/2 \\ 3 - 3J_n/2 & 1 - I_n + J_n/2 \end{bmatrix} \qquad (3)$$

$$\Omega = 2i\omega \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4}$$

and

$$N_k = \frac{Gm}{2r^3}\begin{bmatrix} e^{i\theta_k}(1 - J_{k,n}/2) & 3e^{-i\theta_k} - 3J_{k,n}/2 \\ 3e^{i\theta_k} - 3J_{k,n}/2 & e^{-i\theta_k}(1 - J_{k,n}/2) \end{bmatrix} \tag{5}$$

while $I_{k,n}$ and $J_{k,n}$ can be written as

$$I_{k,n} = \frac{1}{4\sin(\frac{\pi|k|}{n})} \tag{6}$$

$$J_{k,n} = \frac{1}{4\sin^3(\frac{\pi|k|}{n})}\ , \tag{7}$$

drawing upon Kolemen and Vanderbei's matrix notation [3]. Physically, Eq. (3), Eq. (4), and Eq. (5) represent the components of the equation of motion for the ring system, which is derived from Newton's law of gravity [3]. A detailed explanation of this Newtonian system's equation of motion derivation leading to the structure of Eq. (2) resides in Vanderbei and Kolemen's *Linear Stability of Ring Systems*. Based upon the matrix's structure, the necessary solutions are of the form

$$\begin{bmatrix} \delta W_0 \\ \delta W_1 \\ \vdots \\ \delta W_{n-1} \end{bmatrix} = \begin{bmatrix} \xi \\ \rho\xi \\ \vdots \\ \rho^{n-1}\xi \end{bmatrix} . \tag{8}$$

Continuing, the nontrivial eigenvalue solutions of the matrix are ascertained by analyzing the determinant of

$$\det(D + \rho N_1 + \cdots + \rho^{n-1}N_{n-1} + \lambda\Omega - \lambda^2 I) = 0 \tag{9}$$

where the determinant is necessarily equivalent to zero. In rewriting, Eq. (9) can be represented by

$$\det\left(D + \sum_{k=1}^{n-1}\rho^k N_k + \lambda\Omega - \lambda^2 I\right) = 0\ . \tag{10}$$

Using reduction, the summation term is rewritten as

$$\rho^k J_{k,n} = \sum_{k=1}^{n-1}\frac{1}{4}\sum_{k=1}^{n-1}\frac{e^{\frac{2\pi ijk}{n}}}{sin^3\left(\frac{\theta_k}{2}\right)}\frac{1}{4}\sum_{k=1}^{n-1}\frac{\cos(j\theta_k)}{sin^3\left(\frac{\theta_k}{2}\right)} \equiv \tilde{J}_{j,n}\ , \tag{11}$$

which similarly implies the structure of the following summation terms:

$$\textstyle\sum_{k=1}^{n-1}\rho^k e^{i\theta_k}J_{k,n} = \tilde{J}_{j+1,n} \tag{12}$$

$$\textstyle\sum_{k=1}^{n-1}\rho^k e^{-i\theta_k}J_{k,n} = \tilde{J}_{j-1,n}\ . \tag{13}$$

Further manipulation of the summation term in conjunction with the Kronecker delta yields

$$\textstyle\sum_{k=1}^{n-1}\rho^k e^{i\theta_k} = \sum_{k=1}^{n-1}e^{ij\theta_k}e^{i\theta_k} = \sum_{k=1}^{n-1}e^{i(j+1)\theta_k} = \begin{cases} n-1, & j = n-1 \\ -1, & otherwise \end{cases} \tag{14}$$

$$\Omega\textstyle\sum_{k=1}^{n-1}\rho^k e^{-i\theta_k} = \begin{cases} n-1, & j = 1 \\ -1, & otherwise \end{cases} . \tag{15}$$

To explicitly define the summation term, Eq. (5) and Eq. (11)-(15) are substituted into the summation term, yielding

$$\textstyle\sum_{k=1}^{n-1}\rho^k N_k = \frac{Gm}{2r^3} * \begin{bmatrix} -1 + n\delta_{j=n-1} - \frac{1}{2}\tilde{J}_{j+1,n} & -3 + 3n\delta_{j=1} - \frac{3}{2}\tilde{J}_{j,n} \\ -3 + 3n\delta_{j=n-1} - \frac{3}{2}\tilde{J}_{j,n} & -1 + n\delta_{j=1} - \frac{1}{2}\tilde{J}_{j-1,n} \end{bmatrix} \tag{16}$$

where the Kronecker delta, $\delta_{j=k}$, is equivalent to 1 when $j = k$ but 0 for all other values.

Drawing upon Vanderbei and Kolemen's *Linear Stability of Ring Systems*,

$$\omega^2 = \frac{GM}{r^3} + \frac{Gm}{r^3}\textstyle\sum_{k=1}^{n-1}\frac{1}{\sin\left(\frac{\pi k}{n}\right)} = \frac{GM}{r^3} + \frac{Gm}{r^3}I_n \tag{17}$$

in order to satisfy Newton's equations of motion with

$$I_n \equiv \sum_{k=1}^{n-1} \frac{1}{\sin\left(\frac{\pi k}{n}\right)} \quad (18)$$

where ω is the angular velocity of the ring particles orbiting Saturn and the term $I_n$condenses the expression [3].

Next, using the results of Eq. (4), Eq. (6), Eq. (16), Eq. (21), and Eq. (22), the matrix is rewritten as

$$D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I = \begin{bmatrix} \eta_+ & \frac{3}{2}\omega^2 + \frac{3}{2}\alpha_j^2 \\ \frac{3}{2}\omega^2 + \frac{3}{2}\alpha_j^2 & \eta_- \end{bmatrix} \quad (19)$$

where

$$\eta_+ = \frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j+1}^2 - \beta^2 - 2i\omega\lambda - \lambda^2 \quad (20)$$

and

$$\eta_- = \frac{3}{2}\omega^2 + \frac{1}{2}\alpha_{j-1}^2 - \beta^2 + 2i\omega\lambda - \lambda^2 \ . \quad (21)$$

Next, leveraging the analytical software, Mathematica, the determinant of Eq. (10) is

$$\det\left(D + \sum_{k=1}^{n-1} \rho^k N_k + \lambda\Omega - \lambda^2 I\right) = \lambda^4 + \left(\omega^2 - \alpha_{\frac{n}{2\pm1}}^2 + 2\beta^2\right)\lambda^2 + 3\omega^2\left(\frac{1}{2}\alpha_{\frac{n}{2\pm1}}^2 + \frac{3}{2}\alpha_{\frac{n}{2}}^2 - \beta^2\right) + \left(\frac{1}{2}\alpha_{n/2\pm1}^2 - \beta^2\right)^2 - \frac{9}{4}\alpha_{\frac{n}{2}}^4 \quad (22)$$

with eigenvalue

$$\omega^2 \geq 4\alpha_{\frac{n}{2\pm1}}^2 + 9\alpha_{\frac{n}{2}}^2 - 8\beta^2 + \sqrt{\left(4\alpha_{n/2\pm1}^2 - 9\alpha_{\frac{n}{2}}^2 - 8\beta^2\right)^2 - 9\alpha_{\frac{n}{2}}^4} \ . \quad (23)$$

Here, only the greater-than constraint physically relevant so the result is constrained to the positive root [3]. The terms

$$\alpha_j^2 = \frac{Gm}{2r^3}(J_n - \tilde{J}_{j,n}) \quad (24)$$

and

$$\beta^2 = \frac{Gm}{2r^3} I_n \quad (25)$$

are defined to simplify the lengthy expression for the determinant of Eq. (10) [3]. Finally, substituting the values of ω^2, α_j^2, and β^2 into Eq. (23) and rearranging to find M/m, the inequality becomes

$$\frac{M}{m} \geq 2(J_n - \tilde{J}_{\frac{n}{2\pm1},n}) + \frac{9}{2}(J_n - \tilde{J}_{\frac{n}{2},n}) - 5I_n + [(2(J_n - \tilde{J}_{\frac{n}{2\pm1},n}) + \frac{9}{2}(J_n - \tilde{J}_{\frac{n}{2},n}) - 4I_n)^2 - \frac{9}{4}(J_n - \tilde{J}_{\frac{n}{2},n})^2]^{\frac{1}{2}} \ . \quad (26)$$

In other words, the ratio of the large primary mass to the smaller collective mass of *n* finite particles obeys the relationship characterized by Eq. (26), which stems from the eigenvalues of Eq. (22).

Addressing the present case of Saturn's rings, in the limit of large *n* it is clear that $\tilde{J}_{\frac{n}{2}\pm1,n} \approx \tilde{J}_{\frac{n}{2},n}$ and, congruently, $J_n \gg I_n$. Leveraging this approximation, $\tilde{J}_{\frac{n}{2},n}$ can be restated as

$$\tilde{J}_{\frac{n}{2},n} \approx \frac{1}{2}\sum_{k=1}^{\frac{n}{2}} \frac{(-1)^k}{sin^3\left(\frac{k\pi}{n}\right)} \approx \frac{n^3}{2\pi^3}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} = \frac{-3n^3}{8\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3} = \frac{-3}{8}\sum_{k=1}^{\frac{n}{2}} \frac{(-1)^k}{sin^3\left(\frac{k\pi}{n}\right)} \approx -\frac{3}{4}J_n. \quad (27)$$

Next, inserting this value for $\tilde{J}_{\frac{n}{2},n}$ into Eq. (26), the mass ratio reduces to

$$\frac{M_{Saturn}}{m_{rings}} \geq \frac{7}{8}(13 + 4\sqrt{10})\, J_n \ . \quad (28)$$

In order to regain Maxwell's afore noted inequality in Eq. (1) describing the mass ratio of Saturn to the n bodies, Eq. (28) is rearranged to give

$$m_{Rings} \leq \frac{M_{Saturn}}{\frac{7}{8}(13+4\sqrt{10})J_n} \approx \frac{2.298 M_{Saturn}}{n^2}, \qquad (29)$$

representing the mass ratio requirement for ring system stability in the limit of large n. This concludes the mathematical investigation into the stability of Saturn's rings of n equal-mass particles in circular orbit about a large primary mass where $m_{particles} \ll M_{Saturn}$.

IV. RESULTS

A linear stability analysis of a system of n equal mass bodies in circular orbit about a large primary provides insight into the mechanics of Saturn's ring. Notably, a ring of equal-mass particles in circular orbit about a mass much larger than that of the particles remains in stable motion even when affected by small perturbation disturbances. Even further, the system of Saturn's rings parallels n-body satellite systems of vanishingly small masses in rotation about a center of mass point.

From the linear stability analysis, it is concluded that Saturn's rings are stable so long as the following inequality of Eq. (29), $m_{Rings} \leq \frac{2.298 M_{Saturn}}{n^2}$, is not violated. That is to say, a ring of n satellites in circular orbit about a planet is stable against perturbations of the relative positions if the mass of each satellite is vanishingly small in comparison with the total mass of the planet. In addition, the mathematical conclusion that this system of small satellites orbiting a planet is physically stable confirms that Saturn's rings consist of n finite particles where $n \gg 1$ and, by inductive reasoning, does not consist of either a rigid planar disk or a gaseous cloud.

At the core, this stability analysis reveals that a system of n satellites in orbit around a large primary represents a stable ring system. The system remains in stable motion even when affected by in-plane perturbations. With regard to Saturn's specific ring stability, in the limit of large n, the system essentially consists of a very large number of finite, vanishingly small bodies orbiting around a much more massive primary body, the planet. Mathematically, the derivation reveals that so long as the total mass of the n particles obeys the inequality of Eq. (29), which relates the mass of the rings to the mass of the planet, the system is stable.

These results parallel Maxwell's marked conclusion in 1856, yielding the same relationship between the collective mass of the ring bodies and the mass of the planet about which the rings orbit. Attaining the mass ration between Saturn and its ring system characterized by Eq. (29), the stability of a planar ring of n small particles orbiting Saturn in the limit of $n \gg 1$ is proven. This ultimately regains Maxwell's thesis, which proposes that Saturn's rings exist as a system of many non-interacting, finite bodies.

Uncertainty associated with this derivation arose most notably from two primary sources. First, the orbits of the particles are considered to be circular instead of elliptical in order to clarify the derivation and leverage Newton's simplistic gravitational laws. In seeking to understand the composition of Saturn's rings, the shape of the orbit does not significantly affect the determination that the rings are composed of small satellites. However, this approximation limits the ability of these results to accurately predict the nature of the satellites' orbits and their response to perturbations. Centrally, approximating the orbit as circular introduces uncertainty into the rotational dynamics of the elliptical orbits, but this uncertainty is negligible in the context of assessing the physical makeup of the ring system.

A second notable source of uncertainty results from approximating the ring system as planar and ignoring the ring system's measureable thickness. Spacecraft probes reveal that gravitational perturbations caused by nearby masses including Saturn's moons cause the rings to have a local thickness of up to 1 km [5]. Approximating the ring system as planar allows for the mathematical exclusion of instabilities stemming form out-of-plane perturbations, which complicate the derivation without physical significance relevant to this discussion. Out-of-plane perturbations are less destabilizing to the system than in-plane perturbations and, consequently, do not have a noteworthy effect on the final mass ratio result [3]. By restricting the analysis to the planar case, the derivation works in the complex plane, simplifying the analysis without sacrificing the accuracy of the final result.

Overall, the results of this mathematical derivation are sufficiently reliable in that they accurately predict the composition of the rings and provide a physically stable mass ratio describing the n small satellites orbiting Saturn. These results, despite the uncertainty arising from approximations made about the system dynamics, align with both theoretical and experimental data, suggesting that the approximations made do not tangibly affect the mathematical outcomes.

V. DISCUSSION AND CONCLUSION

The theoretical results of this derivation, beginning with a Newtonian analysis and moving towards a linear stability approximation suitably align with Maxwell's 1856 theoretical results and spectroscopic data collected by James Keeler at the Allegheny Observatory in 1895 [6]. Most notably, these theoretical results correctly predict the many-body nature of Saturn's rings, experimentally verified by modern spacecraft travel. To this point, Saturn's ring system is compositionally an asteroid belt consisting of many small, non-connected bodies that range in size from micrometers to meters which orbit around the much more massive primary, Saturn [7].

In order for Saturn's ring system to achieve stability, these theoretical results require that the small bodies rotating around Saturn be non-gaseous, finite masses falling within the mass ratio of Eq. (29). In other words, the ring bodies must be small satellites of mass much smaller than the body they orbit that do not appreciably interact with each other. Further, the theoretical results demand that the number of small bodies rotating around Saturn be in the limit of large n, such where $n \gg 1$. Both spectroscopic analysis using Doppler shift mathematics and space probe photography reveal that Saturn's rings are physically composed of small satellites orbiting the much more massive primary, Saturn [8]. In other words, experimental observations reveal that the rings are not composed of either a singular disk-shaped mass or of a gaseous cloud. Continuing, according to the *Voyager* space probes, the total mass of the rings is approximately $3 * 10^{19}$ kg, representing a fraction of only 50 ppb the mass of Saturn [9].

Altogether, the results of this derivation yield the same conclusion Maxwell arrived at in 1856. Attaining the mass ration between Saturn and its ring system characterized by Eq. (29), the stability of a planar ring of n small particles orbiting Saturn in the limit of $n \gg 1$ is proven. This ultimately regains Maxwell's prize-winning thesis, which defines Saturn's rings as a system of many non-interacting, finite bodies. Moreover, modern spacecraft probes reveal the physical authenticity of these theoretical results, experimentally proving that Saturn's rings consist of many small satellites orbiting the planet in stable motion. Therefore, the results of this mathematical analysis predict the stability of a ring system in the limit $n \gg 1$ for finite masses with sufficient accuracy, aligning with previous mathematical derivations as well as experimental results of space probes.

VI. REFERENCES

[1] J.C. Maxwell, *On the Stability of Motions of Saturn's Rings*, Macmillan and Company, Cambridge, (1859).

[2] G. R. Goldsbrough, *The Stability of Saturn's Rings,* Philosophical Transactions of the Royal Society of London, Series A, Mathematical and Physical Sciences, vol. 244, no. 874, pp. 1–17 (1951).

[3] R. J. Vanderbei and E. Kolemen, *Linear Stability of Ring Systems*, The Astronomical Journal, vol. 133, no. 2, pp. 656 – 664 (2007).

[4] R. Moeckel, *Linear Stability of Relative Equilibria With a Dominant Mass,* Journal of Dynamics and Differential Equations, vol. 6, pp. 37–51 (1994).

[5] Cornell University News Service, *Researchers Find Gravitational Wakes In Saturn's Rings*, ScienceDaily, pp. 12 – 24 (2005).

[6] J. E. Keeler, *A Spectroscopic Proof of the Meteoric Constitution of Saturn's Rings*, Astrophysical Journal, vol. 1, pp. 416 – 427 (1895).

[7] M. S. Tiscareno, "Planetary Rings." *Planets, Stars, and Stellar Systems*, edited by Terry Oswalt et al., Springer pp. 61 – 63 (2013).

[8] H. A. Zebker et al., *Saturn's Rings – Particle Size Distributions for Thin Layer Model*, Icarus, vol. 64, no. 3, pp. 531 – 548 (1985).

[9] J. Brainerd, *Saturn's Rings*, The Astrophysics Spectator, vol. 1, no. 8 (2004).